Problem 2.15

Consider a projectile launched with velocity (v_{xo}, v_{yo}) from horizontal ground (with x measured horizontally and y vertically up). Assuming no air resistance, find how long the projectile is in the air and show that the distance it travels before landing (the horizontal range) is $2v_{xo}v_{yo}/g$.

Solution

Newton's second law gives two equations of motion, one for each dimension the projectile moves in.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \end{cases}$$

Since there's no air resistance, the only force to consider is the one due to gravity.

$$\begin{cases} 0 = ma_x \\ -mg = ma_y \end{cases}$$

$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = v_{xo} \\ \dfrac{dy}{dt} = -gt + v_{yo} \end{cases}$$

$$\begin{cases} x(t) = v_{xo}t + x_o \\ y(t) = -\dfrac{1}{2}gt^2 + v_{yo}t + y_o \end{cases}$$

Take the launching site to be the origin so that $x_o = 0$ and $y_o = 0$.

$$\begin{cases} x(t) = v_{xo}t \\ y(t) = -\dfrac{1}{2}gt^2 + v_{yo}t \end{cases}$$

Now that the equations of motion are solved, every question about the projectile's motion can be answered.

In order to find how long the projectile is in the air, set $y(t_{\text{air}}) = 0$ and solve the equation for t_{air}.

$$0 = -\frac{1}{2}gt_{\text{air}}^2 + v_{y\text{o}}t_{\text{air}}$$

$$0 = \left(-\frac{1}{2}gt_{\text{air}} + v_{y\text{o}}\right)t_{\text{air}}$$

By the zero product property,

$$-\frac{1}{2}gt_{\text{air}} + v_{y\text{o}} = 0 \quad \text{or} \quad t_{\text{air}} = 0$$

$$\boxed{t_{\text{air}} = \frac{2v_{y\text{o}}}{g}} \quad \text{or} \quad t_{\text{air}} = 0.$$

The projectile is launched at $t = 0$, so $t = 2v_{y\text{o}}/g$ must be when it hits the floor. Plug this time into the formula for $x(t)$ to find how far it goes horizontally.

$$x\left(\frac{2v_{y\text{o}}}{g}\right) = v_{x\text{o}}\left(\frac{2v_{y\text{o}}}{g}\right)$$

$$= \frac{2v_{x\text{o}}v_{y\text{o}}}{g}$$

This is the horizontal range.